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The problem of the failure of a dam in two-layer shallow water (linear approximation)☆

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ABSTRACT

The flows that occur in the solution, in the linear approximation, of the problem of the failure of a dam for the model of two-layer shallow water with a free boundary are analysed qualitatively. It is shown that, apart from symmetry, four basic processes and four transients exist. They are distinguished by the type of jumps in the levels on the lines of discontinuity and by the direction of the velocities in the liquid layers. Examples of the profiles of all these flows are presented.

Three differential models of two-layer shallow water have been developed and analysed:[1] Model **I** is a model with a free boundary, Model **II** is an "under a cover" model and Model **III** is the general limiting case of these obtained when the ratio of the densities of the liquid in the layers tends to unity. The hyperbolic domains, in which discontinuous solutions are allowed, have been constructed for all three models and, in this connection, a problem was set up on the formulation of these models in the form of complete systems of conservation laws[2,3] and the study of the stable discontinuous solutions with intermittent waves which are admitted by these systems.

One of the possible approaches to solving this problem, which goes back to the classical one-dimensional models of gas dynamics[4] and one-layer shallow water,[5,6] is associated with the proof of the unique solvability in the domain of hyperbolicity of the problem of the decomposition of an arbitrary discontinuity. This approach has been implemented in the case of the simplest Model **III**.[7] However, it could not be successfully extended to the more complex Models **I** and **II**, since the problems of the decomposition of a discontinuity which arise here are too complex and it turned out to be impossible to carry out a complete analytical analysis of them in the general case.

A criterion for the correctness of the complete system of conservation laws has been proposed,[8] which assumes maximum matching of the domain of convexity of the closing conservation law and the domain of hyperbolicity of the differential model. A correct complete system of conservation laws has been selected on the basis of this criterion for a model of two-layer shallow water with a free upper boundary. The laws of conservation of mass, conservation of total momentum and the discontinuity of the velocity on the interface of the layers appear as the basic conservation laws and the law of conservation of total energy appears as the closing law in this system. The stable intermittent waves admitted by this system have been studied[8] and an extension of this system to the spatially two-dimensional case has been given.

In order to confirm that the proposed complete system of conservation laws correctly reflects the parameters of the discontinuous waves in real flows, it is necessary to carry out a comparative analysis of the solutions of the problem or the decomposition of a discontinuity (and, to begin with, its most important special case, which can be most easily experimentally realized, that is, the problem of the failure of a dam), which are obtained using this system and other complete systems having a definite physical meaning. However, this can only be performed as the result of a numerical experiment. In this paper, the problem is therefore considered in a linear approximation with the aim of carrying out a preliminary qualitative analysis of the possible type of solutions of the problem of the failure of a dam in two-layer shallow water. In this approach, all the complete systems of conservation laws become equivalent and ordinary waves (that is, discontinuous waves propagating at a constant velocity and centred subsiding waves) are replaced by discontinuity lines.

The system of equations of two-layer shallow water with a free boundary is presented in Section 1 and its linear approximation with respect to the constant solution, corresponding to a state of rest, is constructed. In Section 2, the hyperbolicity of the resulting linear system is checked and the invariant form of it is presented. The solution of the Cauchy problem with arbitrary initial data is written out using this invariant form. A problem of the decomposition of a discontinuity, for which the relations between the discontinuities in the flow parameters and the initial discontinuities of the Riemann invariants are obtained, is considered in Section 3 for the linear system constructed. It follows from these relations that, apart from symmetry, ten classes of solutions of the problem exist which have qualitatively

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different profiles of the free surface and the interface between the layers. The problem of the failure of a dam is studied in Sections 4 and 5 which is obtained as a special case of the problem of the decomposition of a discontinuity with the additional condition that the initial velocities in the layers are zero. It is shown that, apart from symmetry, there are four main and four transitional (degenerate) classes of qualitatively different solutions in this problem for which the profiles of the levels of the lower and upper layers are constructed and the domains of their existence in the planes of the different defining parameters are distinguished.

1. Formulation of the problem

The motion of two-layer shallow water over a horizontal bottom with a free boundary surface without taking account of the effect of friction and on the assumption that the gravitational acceleration $g = 1$ is described by the following system of differential equations:[1]

$$\begin{aligned} &H_t + UH_x + HU_x = 0, \quad h_t + uh_x + hu_x = 0 \\ &U_t + UU_x + H_x + \mu h_x = 0, \quad u_t + uu_x + H_x + h_x = 0 \end{aligned} \tag{1.1}$$

where H and U are the depth and velocity of the lower layer, h and u are the depth and velocity of the upper layer, and $\mu = \rho_2/\rho_1 < 1$ is the ratio of the densities ρ_2 and ρ_1 of the upper and lower layers.

We will consider the constant solution of system (1.1)

$$H_0(x,t) = 1, \quad h_0(x,t) = \text{const}, \quad U_0(x,t) = u_0(x,t) = 0 \tag{1.2}$$

and its small perturbation

$$H = 1 + \delta H, \quad h = h_0 + \delta h, \quad U = \delta U, \quad u = \delta u \tag{1.3}$$

in which $\delta f \ll 1$. Substituting the functions (1.3) into system (1.1), only retaining the quantities $O(\delta f)$ in it after this and, for brevity, omitting the symbol δ, we obtain the linear approximation of system (1.1) relative to the state of rest (1.2)

$$\begin{aligned} &H_t + U_x = 0, \quad h_t + h_0 u_x = 0 \\ &U_t + H_x + \mu h_x = 0, \quad u_t + H_x + h_x = 0 \end{aligned} \tag{1.4}$$

We write system (1.4) in the vector form

$$\mathbf{u}_t + A\mathbf{u}_x = 0; \quad \mathbf{u} = \begin{Vmatrix} H \\ h \\ U \\ u \end{Vmatrix}, \quad A = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h_0 \\ 1 & \mu & 0 & 0 \\ 1 & 1 & 0 & 0 \end{Vmatrix} \tag{1.5}$$

and consider for it a problem on the decomposition of the initial discontinuity, that is, a Cauchy problem with the following piecewise-constant initial data

$$\mathbf{u}(x,0) = \mathbf{v}(x) = \begin{cases} \mathbf{v}^l, & x \le 0 \\ \mathbf{v}^r, & x > 0 \end{cases} \quad \mathbf{v}^l, \mathbf{v}^r = \text{const} \tag{1.6}$$

The aim of this paper is to carry out a qualitative analysis of the solution of the problem of the decomposition of the discontinuity (1.5), (1.6) and, to begin with, its most important special case, that is, the solutions of the problem of the failure of a dam which is obtained if, in the initial data (1.6),

$$U^l = U^r = u^l = u^r = 0 \tag{1.7}$$

In the case of the linear system (1.5), this problem admits of a complete analytical investigation, unlike the analogous problem for the quasilinear system (1.1), which can only be investigated to the full extent numerically.

2. Solution of the Cauchy problem

It has been shown[1] that quasilinear system (1.1) is hyperbolic when one of the two inequalities

$$|U - u| < \sqrt{H} f_1(h/H), \quad |U - u| > \sqrt{H} f_2(h/H) \tag{2.1}$$

is satisfied, in which f_1 and f_2 are certain positive functions. Since the constant solution (1.2) satisfies the first equality of (2.1), system (1.1) is hyperbolic for this solution. The hyperbolicity of the linear system (1.4), which, when written in vector form, has the form (1.5), follows from this.

We will now directly verify the hyperbolicity of system (1.5) by finding the roots of the characteristic equation

$$P(\lambda) = |A - \lambda E| = \lambda^4 - r\lambda^2 + \eta h_0 = 0; \quad r = h_0 + 1, \quad \eta = 1 - \mu > 0 \tag{2.2}$$

corresponding to it.

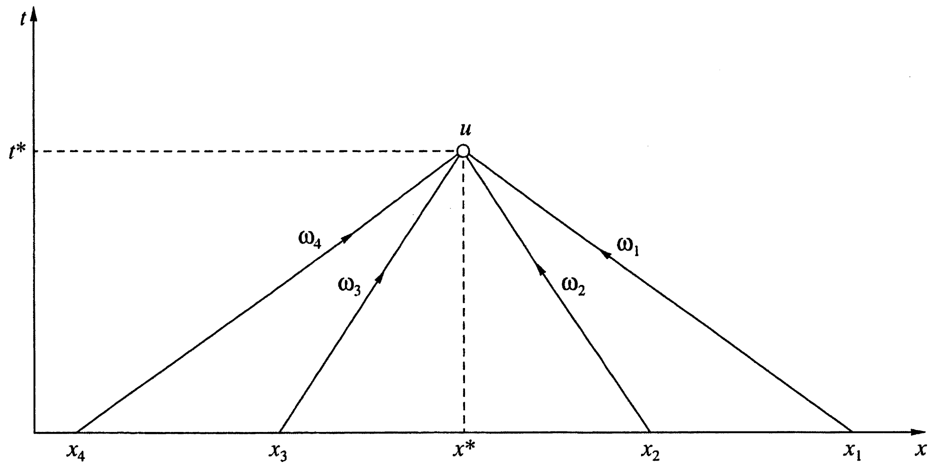

Fig. 1.

Since $P(1) = -\mu h_0 < 0$, the biquadratic equation (2.2) in $y = \lambda^2$ has two different positive roots. Hence, the eigenvalues of the matrix A, which are defined by the formulae

$$\lambda_3 = -\lambda_2 = \sqrt{(r-d)/2}, \quad \lambda_4 = -\lambda_1 = \sqrt{(r+d)/2}; \quad d = \sqrt{r^2 - 4\eta h_0} \tag{2.3}$$

turn out to be real and different.

Since the linear system (1.5) is hyperbolic, it can be written in the invariant form[3]

$$\boldsymbol{\omega}_t + \Lambda\boldsymbol{\omega}_x = 0 \Leftrightarrow (\omega_i)_t + \lambda_i(\omega_i)_x = 0 \tag{2.4}$$

Henceforth, unless otherwise stated, $i = 1, 2, 3, 4$; $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T = S\mathbf{u}$ is the vector of the invariants, $\Lambda = (\lambda_i \delta_j^i)$ is a diagonal matrix and $S = (\mathbf{l}^1, \mathbf{l}^2, \mathbf{l}^3, \mathbf{l}^4)^T$ is a non-degenerate matrix, the rows of which

$$\mathbf{l}^i = (\lambda_i, b_i, 1, a_i), \quad a_i = \lambda_i^2 - 1, \quad b_i = \lambda_i a_i / h_0 = (\lambda_i^2 - \eta)/\lambda_i, \tag{2.5}$$

are the left eigenvectors of matrix A, that is,

$$\mathbf{l}^i A = \lambda_i \mathbf{l}^i \Leftrightarrow SA = \Lambda S \tag{2.6}$$

It follows from relations (2.6) that the matrix $S^{-1} = (\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3, \mathbf{r}^4)$, which is the inverse of S, satisfies the relation

$$AS^{-1} = S^{-1}\Lambda \Leftrightarrow A\mathbf{r}^i = \lambda_i \mathbf{r}^i$$

by virtue of which its columns

$$\mathbf{r}^i = \frac{1}{\rho_i}(\mu, a_i, \mu\lambda_i, b_i), \quad \rho_i = 2(\mu\lambda_i + a_i b_i) = 2\lambda_i\left(\mu + \frac{a_i^2}{h_0}\right) \tag{2.7}$$

are the right eigenvectors of the matrix A.

Using the invariant form (2.4) of writing the linear system (1.5), the solution of the Cauchy problem $\mathbf{u}(x, 0) = \boldsymbol{\upsilon}(x)$ for it with an arbitrary initial function $\upsilon(x)$ can be represented in the form[2,3]

$$u_i(x^*, t^*) = (S^{-1}\boldsymbol{\omega}(x^*, t^*))_i = \sum_{k=1}^{4} r_i^k \omega_k(x^*, t^*) \tag{2.8}$$

The invariant $\omega_k(x^*, t^*)$ remains constant along the characteristics $x = \lambda_k t + x_k$ emerging from the point $x_k = x^* - \lambda_k t^*$ on the x axis (see Fig. 1) by virtue of which

$$\omega_k(x^*, t^*) = \omega_k(x_k, 0) = \mathbf{l}^k \boldsymbol{\upsilon}(x_k) = \sum_{j=1}^{4} l_j^k \upsilon_j(x_k) \tag{2.9}$$

Omitting, for brevity, the arguments x^* and t^*, solution (2.8), when account is taken of the expression for the vector u and the representation (2.7), can be rewritten in the form

$$H = \mu\sum_{k=1}^{4}\frac{\omega_k}{\rho_k}, \quad h = \sum_{k=1}^{4}\frac{a_k\omega_k}{\rho_k}, \quad U = \mu\sum_{k=1}^{4}\frac{\lambda_k\omega_k}{\rho_k}, \quad u = \sum_{k=1}^{4}\frac{b_k\omega_k}{\rho_k} \tag{2.10}$$

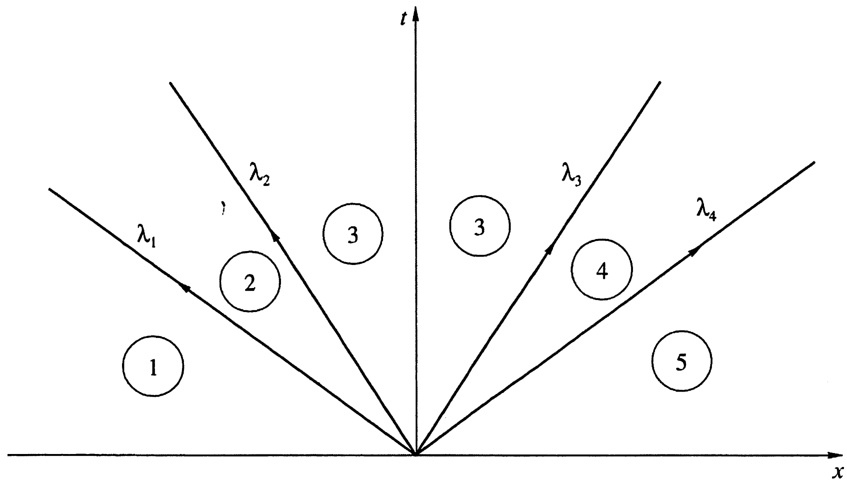

Fig. 2.

It follows from the first two relations of (2.10) that the level of the free surface is calculated using the formula

$$z = H + h = \sum_{k=1}^{4} \frac{c_k \omega_k}{\rho_k}, \quad c_k = a_k + \mu \tag{2.11}$$

3. The problem of the decomposition of a discontinuity

In the case of the problem of the decomposition of discontinuity (1.6), solution (2.8), (2.9) corresponds to four strong discontinuities which propagate along the rays $x = \lambda_i t$, $t < 0$. These discontinuities are connected by the constant flow domains $\mathbf{u}_k$ ($k = 1, 2, \ldots, 5$) labelled with the numbers $k = 1, 2, \ldots, 5$ in Fig. 2 and, at the same time, $\mathbf{u}_1 = \upsilon_l$, $\mathbf{u}_5 = \upsilon_r$. We will now introduce the notation $[\mathbf{u}]_i = \mathbf{u}_{i+1} - \mathbf{u}_i$ for the jump in the solution $\mathbf{u}$ on passing across the i-th discontinuity line $x = \lambda_k t$ from domain i to domain $i + 1$.

Definition 1. The two solutions, $\mathbf{u}$ and $\mathbf{u}'$, of the problem of the decomposition of discontinuity (1.6) have similar profiles if, at each of the four discontinuities $x = \lambda_i t$, the jumps in the depth of the lower layer H and the level of the free surface z occur in the same direction in these solutions, that is, if

$$[H]_i[H']_i > 0, \quad [z]_i[z']_i > 0 \tag{3.1}$$

If the solutions $\mathbf{u}$ and $\mathbf{u}'$ do not satisfy condition (3.1), they have qualitatively different profiles.

We will now construct all the qualitatively different profiles which can be obtained when solving the problem of the decomposition of discontinuity (1.6). The following theorem is required to do this, in the formulation of which, for brevity, the abbreviation $\varphi \sim \psi$ is used for $\mathrm{sign}(\varphi) = \mathrm{sign}(\psi)$.

Theorem 1. *When solving the problem of the decomposition of discontinuity (1.6), the relations*

$$[H]_1 \sim [h]_1 \sim [z]_1 \sim -[\omega_1], \quad [H]_4 \sim [h]_4 \sim [z]_4 \sim [\omega_4] \tag{3.2}$$

$$[H]_2 \sim -[h]_2 \sim -[z]_2 \sim -[\omega_2], \quad [H]_3 \sim -[h]_3 \sim -[z]_3 \sim [\omega_3] \tag{3.3}$$

in which $[\omega_i] = \omega_i^r - \omega_i^l$, where $\omega^r = S\boldsymbol{\upsilon}^r$, $\omega^l = S\boldsymbol{\upsilon}^l$ are the initial values of the vector of the invariants $\boldsymbol{\omega}$, are satisfied for jumps in the depths in the layers and for the jumps in the level of the free surface.

Proof. Since, when solving the problem of the decomposition of discontinuity (1.6) for linear system (1.5), each invariant ω_i only undergoes a discontinuity on passing through the i-th discontinuity line $x = \lambda_i t$, while remaining constant on passing across the remaining discontinuity lines $x = \lambda_k t$, $k \neq i$, the relations

$$[\omega_i]_k = [\omega_i]\delta_k^i = \begin{cases} [\omega_i], & i = k \\ 0, & i \neq k \end{cases} \tag{3.4}$$

hold, where $[\omega_i] = [\omega_i]|_{x=t=0} = \omega_i^r - \omega_i^l$ is the initial jump in the i-th invariant when $x = 0$. Taking account of this, from the first two formulae of (2.10), we obtain

$$[H]_i = \frac{\mu}{\rho_i}[\omega_i], \quad [h]_i = \frac{a_i}{\rho_i}[\omega_i] \tag{3.5}$$

Since, when account is taken of expressions (2.3), the quantities a_i and ρ_i, which are given by formulae (2.5) and (2.7), satisfy the conditions

$$a_1 = a_4 > 0, \quad a_2 = a_3 < 0, \quad \rho_1 = -\rho_4 < 0, \quad \rho_2 = -\rho_3 < 0 \tag{3.6}$$

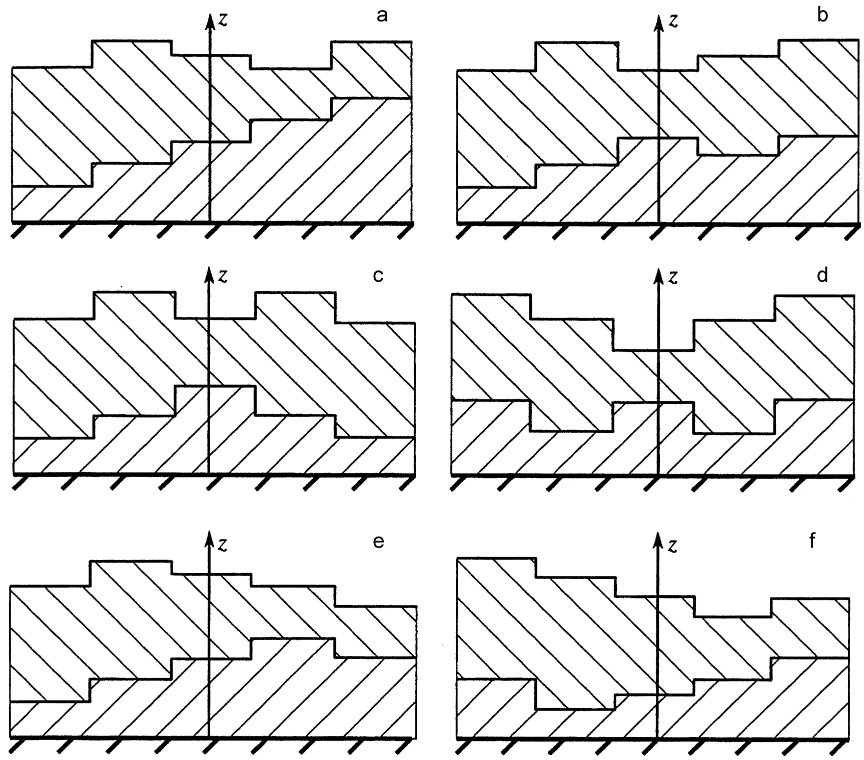

Fig. 3.

then, from relations (3.5), we have

$$[H]_1 \sim [h]_1 \sim -[\omega_1], \quad [H]_4 \sim [h]_4 \sim [\omega_4]$$
$$[H]_2 \sim -[h]_2 \sim -[\omega_2], \quad [H]_3 \sim -[h]_3 \sim [\omega_3] \tag{3.7}$$

We now note that each coefficient $c_k = a_k + \mu$ in formula (2.11) has the same sign as the coefficient a_k in the second formula of (2.10). When $k = 1, 4$, this is obvious by virtue of the positiveness of the quantities μ and $a_1 = a_4$, and, when $k = 2, 3$ and account is taken of relations (2.3) and (2.4), it follows from the chain of inequalities

$$a_k + \mu = \lambda_k^2 - \eta < 0 \Leftrightarrow (r - 2\eta)^2 < d^2 \Leftrightarrow \eta < 1$$

Hence, $c_i \sim a_i \Rightarrow [z]_i \sim [h]_i$, and the theorem is proved.

It follows from this theorem that, when solving the problem of the decomposition of discontinuity (1.6), the directions of the jumps in the depths and levels are completely defined by the signs of the jumps in the four invariants. This distinguishes sixteen qualitatively different profiles, six of which are shown in Fig. 3. The remaining profiles are obtained in the following manner: a further four profiles are obtained by permutation of the profiles of the lower and upper layers in Fig. 3, *a–d* and the remaining six profiles are obtained by reflections in the *z* axis of those ten solutions which have been constructed that are asymmetric about the *z* axis. It follows from Theorem 1 that the characteristic feature of the profiles constructed is that the jumps in the levels of the lower layer and the free surface in the external discontinuity lines $x = \pm\lambda_1 t$ occur in the same direction and the jumps on the internal discontinuity lines $x = \pm\lambda_2 t$ occur in opposite directions.

The following theorem is proved in a similar manner to Theorem 1 using the last two formulae of (2.10) and taking account of the inequalities

$$\frac{\lambda_1}{\rho_1} = \frac{\lambda_4}{\rho_4} > 0, \quad \frac{\lambda_2}{\rho_2} = \frac{\lambda_3}{\rho_3} > 0, \quad \frac{b_1}{\rho_1} = \frac{b_4}{\rho_4} > 0, \quad \frac{b_2}{\rho_2} = \frac{b_3}{\rho_3} < 0 \tag{3.8}$$

Theorem 2. *The relations*

$$[U]_j \sim [u]_j \sim [\omega_j], \quad j = 1, 4; \quad [U]_j \sim -[u]_j \sim [\omega_j], \quad j = 2, 3 \tag{3.9}$$

are satisfied for the jumps in the velocities in the layers in the solution of the problem of the decomposition of discontinuity (1.6).

It follows from this theorem that the jumps in the velocities on the external discontinuity lines $x = \pm\lambda_1 t$ in the lower and upper layers occur in the same direction and that the jumps on the internal discontinuity lines $x = \pm\lambda_2 t$ occur in opposite directions.

4. The problem of dam failure

We will now consider in greater detail an important special case of the problem of the decomposition of the discontinuity (1.6), the problem of dam failure which is obtained in the case of the zero initial velocities (1.7).

Theorem 3. *When solving the problem of dam failure (1.6), (1.7), the following relations are satisfied*

$$[H]_m = [H]_{5-m} = f_m,\ [h]_m = [h]_{5-m} = \frac{a_1}{\mu} f_m,\ [z]_m = [z]_{5-m} = \frac{c_1}{\mu} f_m;\ m = 1, 2 \tag{4.1}$$

in which

$$f_m = f_m(\alpha, \beta) = \frac{\mu\lambda_m}{\rho_m}\left(\alpha + \frac{a_m}{h_0}\beta\right),\quad m = 1, 2 \tag{4.2}$$

where $\alpha = H^r - H^l$ *and* $\beta = h^r - h^l$ *are the amplitudes of the initial discontinuities in the depths in the lower and upper layers, in the case of the jumps in the depths in the layers and the jumps in the level of the free surface.*

Proof. Taking account of relations (1.6), (1.7) and (2.5), from formula (2.9) we obtain expressions for the initial values of the invariants

$$\omega_k^l = l_1^k \upsilon_1^l + l_2^k \upsilon_2^l = \lambda_k H^l + b_k h^l,\quad \omega_k^r = l_1^k \upsilon_1^r + l_2^k \upsilon_2^r = \lambda_k H^r + b_k h^r \tag{4.3}$$

and, from this, we have

$$[\omega_k] = \omega_k^r - \omega_k^l = \lambda_k(H^r - H^l) + b_k(h^r - h^l) = \lambda_k\left(\alpha + \frac{a_k}{h_0}\beta\right) \tag{4.4}$$

Substituting these values of $[\omega_k]$ into the first formula of (3.5), we obtain

$$[H]_i = \frac{\mu\lambda_i}{\rho_i}\left(\alpha + \frac{a_i}{h_0}\beta\right) = f_i(\alpha, \beta) \tag{4.5}$$

The functions f_3 and f_4 are defined by the equalities $f_1 = f_4$ and $f_2 = f_3$. Taking account of the symmetry conditions (3.6) and (3.8), we find

$$[H]_1 = [H]_4 = f_1(\alpha, \beta),\quad [H]_2 = [H]_3 = f_2(\alpha, \beta) \tag{4.6}$$

The remaining equalities, appearing in formulae (4.1), follow from the relations

$$[h]_i = \frac{a_i}{\mu}[H]_i,\quad [z]_i = \frac{c_i}{\mu}[H]_i \tag{4.7}$$

It follows from this theorem that, when solving the dam failure problem, the jumps in the depths and levels are identical on the external $x = \pm\lambda_1 t$ and the internal $x = \pm\lambda_2 t$ discontinuity lines which considerably restricts the number of qualitatively different profiles of the admissible solutions shown in Fig. 3, by distinguishing from them just four profiles which, apart from the symmetry about the z axis and the permutation of the profiles of the lower and upper layers, are shown in Fig. 3,*a*. The domains of existence of flows with these profiles in the plane of the variables α and β are shown in Fig. 4, *a*. In domains **I** and **II** which are obtained when $f_1 > 0, f_2 > 0$ and $f_1 > 0, f_2 < 0$, a flow of the type shown in Fig. 3, *a* and the flow which is obtained from it by permutation of the profiles of the layers respectively are realized but, in domains **I**′ and **II**′ which are obtained when $f_1 < 0, f_2 < 0$ and $f_1 < 0, f_2 > 0$, flows occur which are symmetrical to them about the z axis. If the values of α and β lie on one of the lines $f_1 = 0$ or $f_2 = 0$, then transitional flows occur in which only one of a pair of symmetrical discontinuities remains. When $f_1(\alpha, \beta) = 0$, the external discontinuities located on the lines $x = \pm\lambda_1 t$ degenerate and, when $f_2(\alpha, \beta) = 0$, the internal discontinuities located on the lines $x = \pm\lambda_2 t$ degenerate.

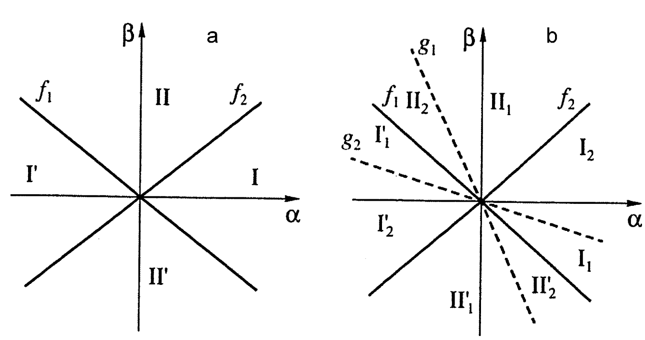

Fig. 4.

The following theorem is proved in a similar manner to Theorem 3 using formulae (2.10).

Theorem 4. *When solving the dam failure problem (1.6), (1.7), the following relations are satisfied*

$$[U]_m = -[U]_{5-m} = \lambda_m f_m, \quad [u]_m = -[u]_{5-m} = \frac{b_1}{\mu} f_m; \quad m = 1, 2 \tag{4.8}$$

in which the functions f_m are specified using formula (4.2).

This theorem sets constraints on the magnitudes of the jumps in the velocities on the discontinuity lines $x = \lambda_i t$. However, in a qualitative analysis of the resulting solutions, an analysis of the possible directions of the velocities of the liquid in the domains of the constant flows marked with the number 1, 2, . . ., 5 in Fig. 2 is of great interest. We investigate this in the next section.

5. The directions of the velocities in the layers

We first give the following definitions.

Definition 2. The two solutions, **u** and **u**′, of the problem of the decomposition of discontinuity (1.6) have similar velocity fields if, in these solutions, the directions of the velocities are the same in both layers of all domains of the constant flow (see Fig. 2), that is, if

$$U_k U'_k > 0, \quad u_k u'_k > 0; \quad k = 1, 2, \ldots, 5 \tag{5.1}$$

If the solutions **u** and **u**′ do not satisfy condition (5.1), they have qualitatively different velocity fields.

Definition 3. The two solutions, **u** and **u**′, of the problem of the decomposition of discontinuity (1.6) are similar if they have similar profiles and velocity fields, that is, if they simultaneously satisfy conditions (3.1) and (5.1). The solutions **u** and **u**′ are qualitatively different if they are not similar, that is, if they do not satisfy one of the conditions (3.1) or (5.1).

We will now distinguish all the qualitatively different solutions of the dam failure problem. For this purpose we require the following theorem, in formulating which account has been taken of the fact that, in the problem in question,

$$U_1 = U^l = u_1 = u^l = U_5 = U^r = u_5 = u^r = 0 \tag{5.2}$$

Theorem 5. *When solving the dam failure problem (1.6), (1.7) for the velocities in the constant flow domains 2–4, the following relations are satisfied*

$$U_2 = U_4 = \lambda_1 f_1, \quad u_2 = u_4 = \frac{b_1}{\mu} f_1, \quad U_3 = g_1, \quad u_3 = g_2 \tag{5.3}$$

in which

$$g_1 = \lambda_1 f_1 + \lambda_2 f_2 = \mu(a\alpha + b\beta), \quad g_2 = \frac{1}{\mu}(b_1 f_1 + b_2 f_2) = b\alpha + c\beta \tag{5.4}$$

$$a = \frac{\lambda_1^2}{\rho_1} + \frac{\lambda_2^2}{\rho_2} < 0, \quad b = \frac{1}{h_0}\left(\frac{\lambda_1^2 a_1}{\rho_1} + \frac{\lambda_2^2 a_2}{\rho_2}\right) < 0, \quad c = \frac{1}{h_0^2}\left(\frac{\lambda_1^2 a_1^2}{\rho_1} + \frac{\lambda_2^2 a_2^2}{\rho_2}\right) < 0 \tag{5.5}$$

Proof. We first separate the relations for the velocities of the lower layer U_i. Taking account of equalities (5.2), from the first formula of (4.9) we obtain

$$U_2 = U_2 - U_1 = [U_1] = \lambda_1 f_1, \quad U_4 = U_4 - U_5 = -[U_4] = \lambda_1 f_1$$

and, from this, when account is taken of the first formula of (4.10), we have

$$U_3 = U_2 + [U]_2 = U_4 - [U]_3 = \lambda_1 f_1 + \lambda_2 f_2$$

Substituting the representation of the functions f_k in the form (4.2) into these equalities, we arrive at the first relation of (5.4). The expressions for the velocities of the upper layer u_i are obtained in a similar manner. The first and last inequalities of (5.5) follow from the symmetry conditions (2.3) and (3.6) and the second inequality, when account is taken of relations (2.2), (2.3), (2.5) and (2.7), is obtained from the chain of formulae

$$\begin{aligned} b &\sim a_1\lambda_1^2\rho_2 + a_2\lambda_2^2\rho_1 \sim (\lambda_1^2 - \eta)\lambda_2(\rho\lambda_2^2 + \gamma) + (\lambda_2^2 - \eta)\lambda_1(\rho\lambda_1^2 + \gamma) = \\ &= \lambda_2(\gamma\lambda_1^2 - \eta\rho\lambda_2^2 + \theta) + \lambda_1(\gamma\lambda_2^2 - \eta\rho\lambda_1^2 + \theta) = \\ &= (\lambda_1 + \lambda_2)(\gamma\lambda_1\lambda_2 - \eta\rho(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2) + \theta) \sim \eta\rho r - \theta - (\gamma + \eta\rho)\sqrt{\eta h_0} = \\ &= \mu(2\eta h_0 - r\sqrt{\eta h_0}) \sim 4\eta h_0 - r^2 = -(\rho^2 + 4\mu h_0) < 0 \end{aligned}$$

where

$$\rho = h_0 - 1, \quad \gamma = 1 + (\mu - \eta)h_0, \quad \theta = \eta(\rho h_0 - \gamma) = \eta(h_0^2 - 2\mu h_0 - 1)$$

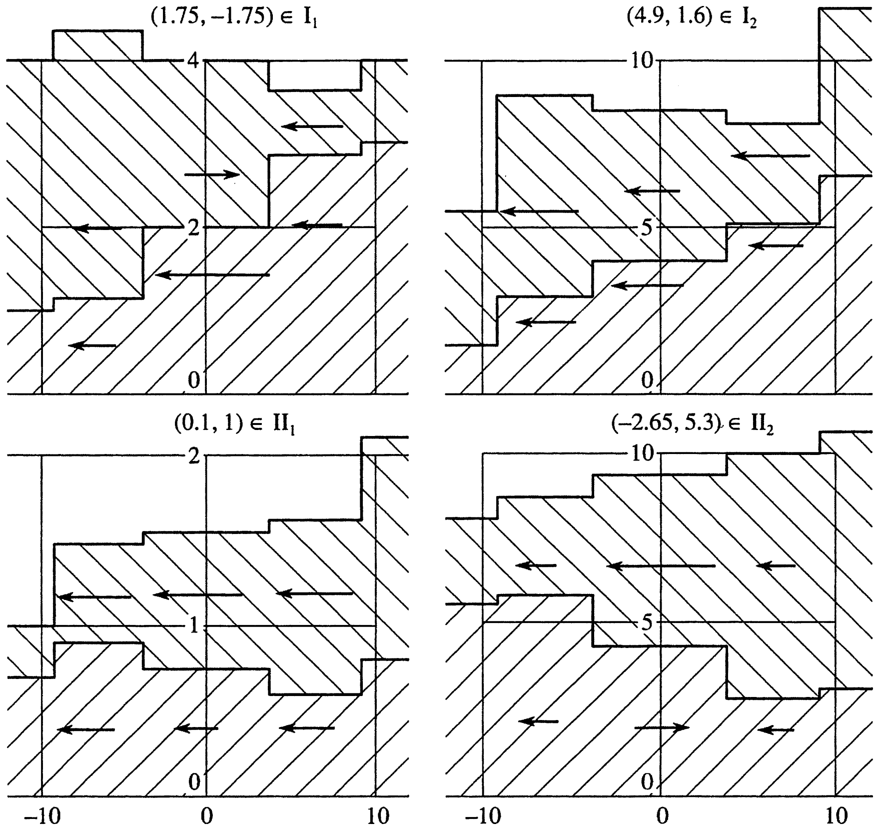

Fig. 5.

When account is taken of Theorem 3, it follows from this theorem that, when solving the dam failure problem (1.6), (1.7), the flow velocities in domains 2 and 4 in Fig. 2 are opposite in sign to the jumps in the depths on the external lines of discontinuities $x = \pm\lambda_1 t$, that is,

$$U_i[H]_i = \lambda_1 f_1^2 < 0, \quad u_i[h]_i = \frac{\lambda_1 a_1^2}{h_0} f_1^2 < 0, \quad i = 1, 4 \tag{5.6}$$

In the central domain 3 in Fig. 2, the directions of the velocities in the layers are determined by the signs of the functions g_1 and g_2, the level lines of which $g_1(\alpha, \beta) = 0$ and $g_2(\alpha, \beta) = 0$, when account is taken of formulae (5.4) and inequalities (5.5), are the strictly monotonically decreasing linear functions $\beta_1(\alpha)$ and $\beta_2(\alpha)$. The graph of the first of these passes through the domains **II** and **II**′ in Fig. 4, and the graph of the second passes through the domains **I** and **I**′. As a result, each of these domains is divided into two subdomains which are denoted by the subscripts 1 and 2 in Fig. 4, *b*. Qualitatively different classes of solutions in the sense of Definition 3 are obtained in all these subdomains. Just eight of these classes are obtained and, apart from the symmetry about the *z* axis, just four.

Examples of the main flow processes, obtained for different initial values of the discontinuities in the depths α and β are shown in Fig. 5. The transitional flow regimes, obtained for the initial values (2.65, −2.8), (1.3, 2.5), (0.2, 1.1) and (4.7, 1.7), which lie on the lines $f_1 = 0$, $f_2 = 0$, $g_1 = 0$ and $g_2 = 0$ respectively (see Fig. 4, *b*), are shown in Fig. 6. When $g_1 = 0$ in the central domain 3 (see Fig. 2), the flow velocity in the lower layer vanishes and, when $g_2 = 0$, the flow velocity in the upper layer vanishes. The length of the arrows in Figs. 5 and 6, by means of which the direction of flow in the layer is indicated, is proportional to the modulus of the velocity of this flow. The solutions shown in Figs. 5 and 6 were obtained for $h_0 = 1$, $\mu = 0.5$, $t = 7$.

It follows from Fig. 4 that the class of solution obtained is completely determined by the angle of inclination φ of the initial vector (α, β) to the α axis. Here, a solution of class $\mathbf{I}_1$ is obtained when $\varphi \in (\varphi_1, \varphi_2)$, of class $\mathbf{I}_2$ when $\varphi \in (\varphi_2, \varphi_3)$, of class $\mathbf{II}_1$ when $\varphi \in (\varphi_3, \varphi_4)$ and of class $\mathbf{II}_2$ when $\varphi \in (\varphi_4, \varphi_5)$ where, when account is taken of formulae (4.2) and (5.4),

$$\varphi_j = \begin{cases} \arcsin k_j, & j = 1, 2, 3 \\ \arccos k_j, & j = 4, 5 \end{cases}$$

$$k_1 = -\frac{h_0}{r_1}, \quad k_2 = \frac{b}{R_1}, \quad k_3 = \frac{h_0}{r_2}, \quad k_4 = \frac{b}{R_2}, \quad k_5 = -\frac{a_1}{r_1}$$

$$r_i = \sqrt{h_0^2 + a_i^2}, \quad i = 1, 2, \quad R_1 = \sqrt{b^2 + c^2}, \quad R_2 = \sqrt{a^2 + b^2} \tag{5.7}$$

Since, when relations (2.3), (2.5), (5.5) and (5.7) are taken into account, the angles φ_j can be considered as functions of the initial depth h_0 and the ratio of the densities μ, that is, $\varphi_j = \varphi_j(h_0, \mu)$, this enables us to construct the domains of existence of the qualitatively different

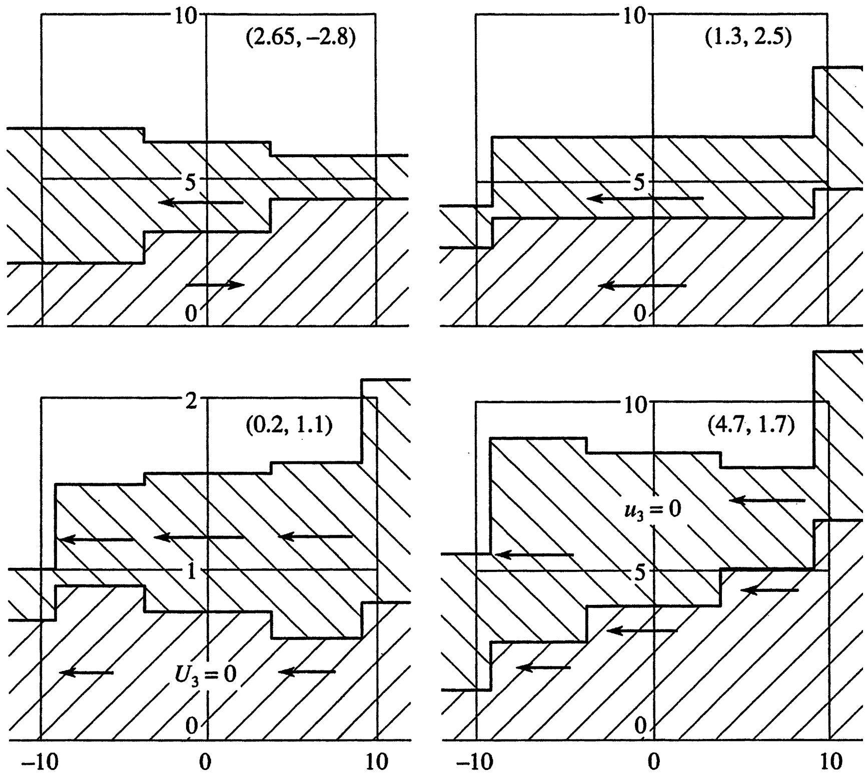

Fig. 6.

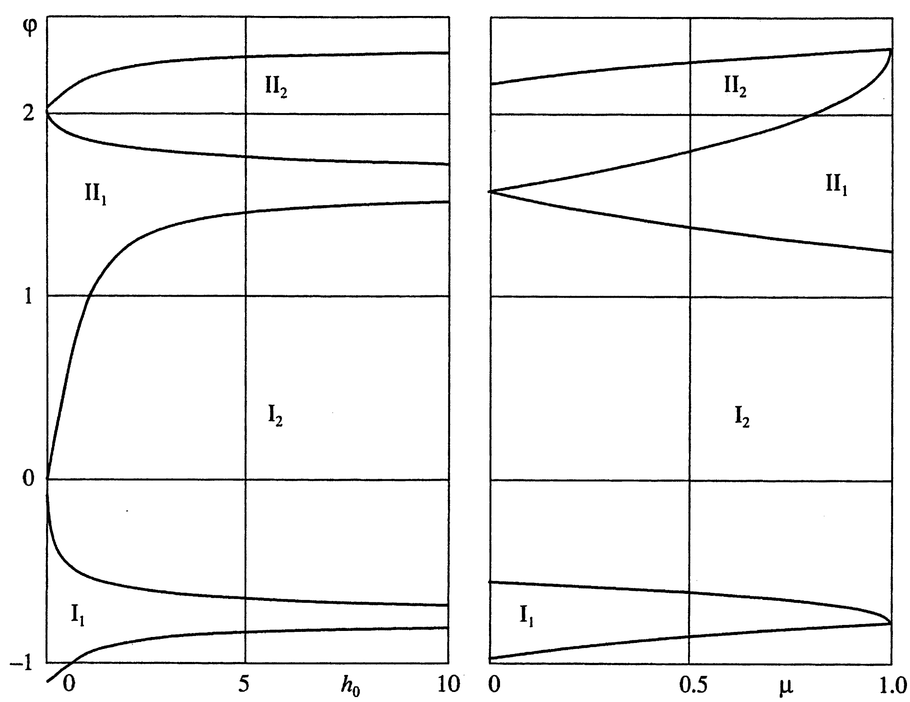

Fig. 7.

classes of solutions of the dam failure problem in the plane of the variables φ, h_0 for fixed μ and in the plane of the variables φ and μ for fixed h_0 (see Fig. 7).

6. Conclusion

A theorem concerning the unique solvability of the problem of the decomposition of the discontinuity (1.6) in the small, that is, for an initial discontinuity of sufficiently small amplitude $|v^l - v^r| = \varepsilon \ll 1$, has been proved for an arbitrary hyperbolic system of conservation laws. The basic shortcoming of this theorem lies in the fact that it does not give an explicit algorithm for constructing the corresponding self-similar solution. At the same time, a problem of the decomposition of a discontinuity for the linear approximation of the initial quasilinear hyperbolic system is obtained in the first approximation with respect to the parameter ε. It is precisely this problem for the system of equations of two-layer shallow water which has been considered in this paper. In the second approximation with respect to the parameter ε, the discontinuities, obtained in the first approximation, separate into rarefaction waves and stable shock waves, the Hugoniot conditions in which will depend on the specific form of the basic conservation laws.

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